Machine Learning – Lecture 3

Probability Density Estimation II

26.04.2016

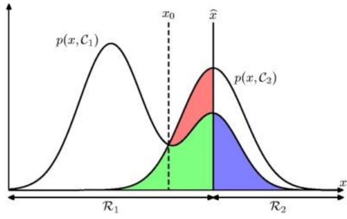
Bastian Leibe
RWTH Aachen
http://www.vision.rwth-aachen.de

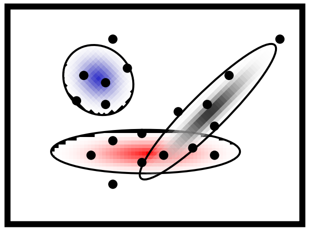
leibe@vision.rwth-aachen.de

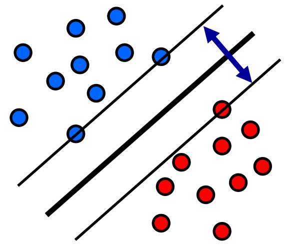
Many slides adapted from B. Schiele

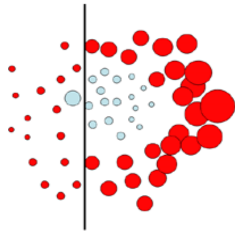
Course Outline

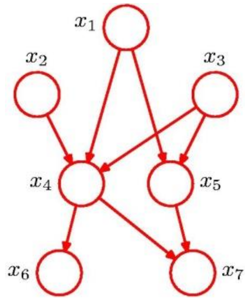
- **Fundamentals (2 weeks)**
  - Bayes Decision Theory
  - Probability Density Estimation
- **Discriminative Approaches (5 weeks)**
  - Linear Discriminant Functions
  - Support Vector Machines
  - Ensemble Methods & Boosting
  - Randomized Trees, Forests & Ferns
- **Generative Models (4 weeks)**
  - Bayesian Networks
  - Markov Random Fields

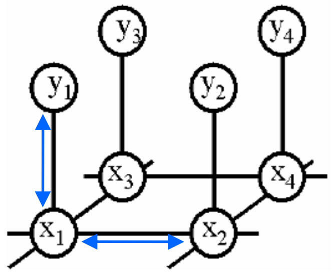












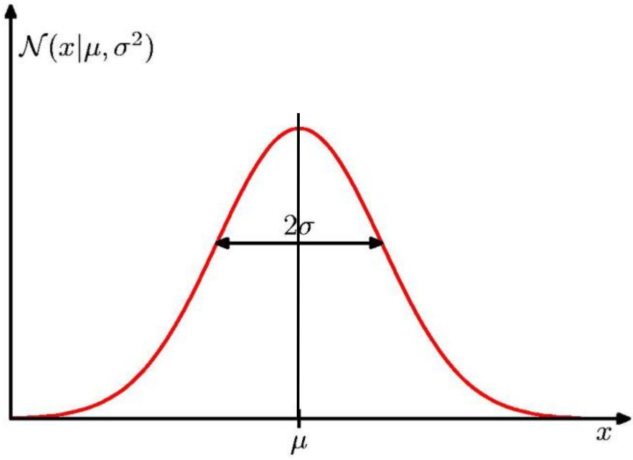
Topics of This Lecture

- Recap: Parametric Methods
  - Maximum Likelihood approach
  - Bayesian Learning
- Non-Parametric Methods
  - Histograms
  - Kernel density estimation
  - K-Nearest Neighbors
  - k-NN for Classification
  - Bias-Variance tradeoff
- Mixture distributions
  - Mixture of Gaussians (MoG)
  - Maximum Likelihood estimation attempt

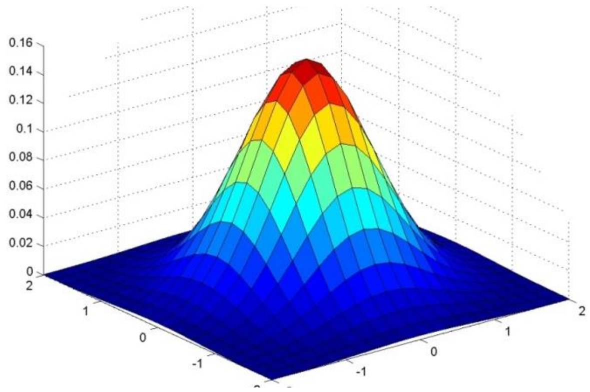
Recap: Gaussian (or Normal) Distribution

- **One-dimensional case**
  - Mean μ
  - Variance σ^2

$$\mathcal{N}(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$$



- **Multi-dimensional case**
  - Mean $\boldsymbol{\mu}$
  - Covariance Σ



$$\mathcal{N}(\mathbf{x}|\boldsymbol{\mu},\boldsymbol{\Sigma}) = \frac{1}{(2\pi)^{D/2}|\boldsymbol{\Sigma}|^{1/2}}\exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right\}$$

Image source: C.M. Bishop, 2006

Recap: Maximum Likelihood Approach

- **Computation of the likelihood**
  - Single data point: $p(x_n|\theta)$
  - Assumption: all data points $X = \{x_1, \dots, x_n\}$ are independent

$$L(\theta) = p(X|\theta) = \prod_{n=1}^{N} p(x_n|\theta)$$

  - Log-likelihood

$$E(\theta) = -\ln L(\theta) = -\sum_{n=1}^{N} \ln p(x_n|\theta)$$

- **Estimation of the parameters θ (Learning)**
  - Maximize the likelihood (=minimize the negative log-likelihood)

$\Rightarrow$ Take the derivative and set it to zero.

$$\frac{\partial}{\partial\theta}E(\theta) = -\sum_{n=1}^{N} \frac{\frac{\partial}{\partial\theta}p(x_n|\theta)}{p(x_n|\theta)} \overset{!}{=} 0$$

Recap: Bayesian Learning Approach

- **Bayesian view:**
  - Consider the parameter vector θ as a random variable.
  - When estimating the parameters, what we compute is

$$p(x|X) = \int p(x,\theta|X)d\theta$$

Assumption: given θ, this doesn't depend on X anymore

$$p(x,\theta|X) = p(x|\theta,\cancel{X})p(\theta|X)$$

$$p(x|X) = \int \underbrace{p(x|\theta)}p(\theta|X)d\theta$$

This is entirely determined by the parameter θ (i.e. by the parametric form of the pdf).

RWTH AACHEN UNIVERSITY

Bayesian Learning Approach

- Discussion

Likelihood of the parametric form θ given the data set X.

Estimate for x based on parametric form θ

Prior for the parameters θ

$$p(x|X) = \int \frac{p(x|\theta)L(\theta)p(\theta)}{\underbrace{\int L(\theta)p(\theta)d\theta}}d\theta$$

Normalization: integrate over all possible values of θ

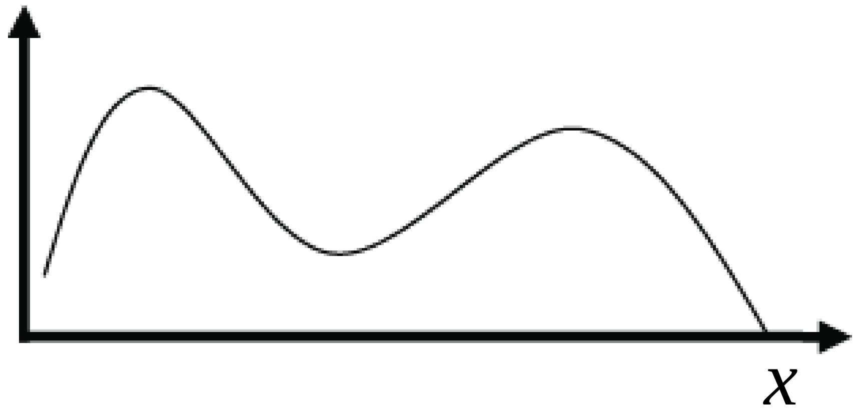
  - If we now plug in a (suitable) prior $p(\theta)$, we can estimate $p(x|X)$ from the data set X.

Topics of This Lecture

- Recap: Bayes Decision Theory
- Parametric Methods
  - Recap: Maximum Likelihood approach
  - Bayesian Learning
- **Non-Parametric Methods**
  - **Histograms**
  - **Kernel density estimation**
  - **K-Nearest Neighbors**
  - **k-NN for Classification**
  - **Bias-Variance tradeoff**
- Mixture distributions
  - Mixture of Gaussians (MoG)
  - Maximum Likelihood estimation attempt

Non-Parametric Methods

- Non-parametric representations
  - Often the functional form of the distribution is unknown



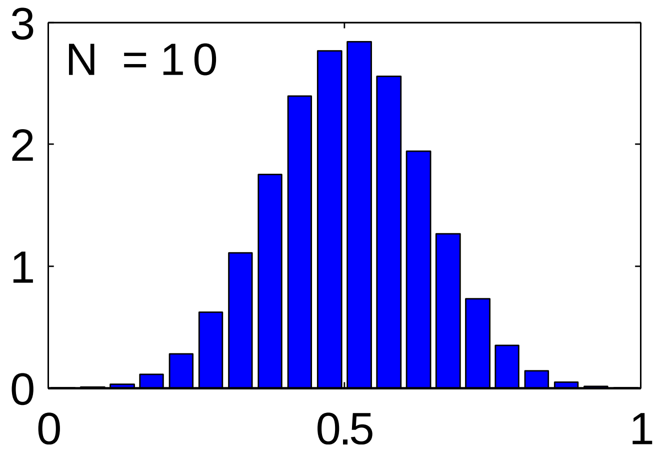
- Estimate probability density from data
  - Histograms
  - Kernel density estimation (Parzen window / Gaussian kernels)
  - k-Nearest-Neighbor

RWTH AACHEN UNIVERSITY

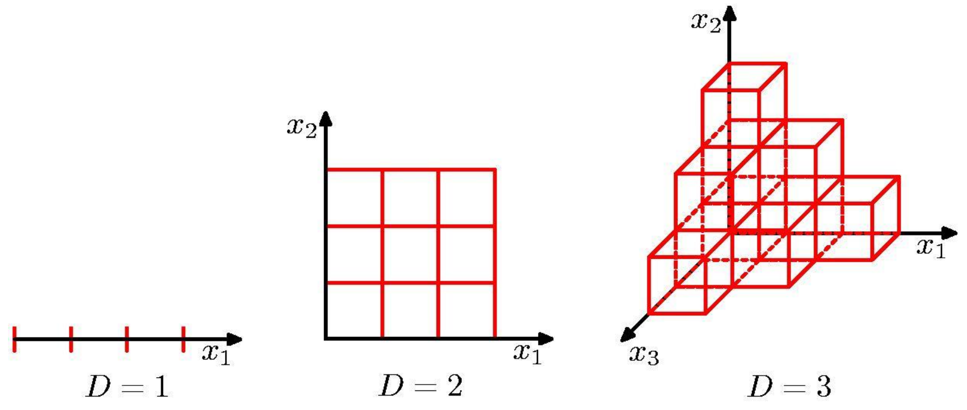
Histograms

- **Basic idea:**
  - **Partition the data space into distinct bins with widths Δ_i and count the number of observations, n_i, in each bin.**

$$p_i = \frac{n_i}{N\Delta_i}$$



  - **Often, the same width is used for all bins, $\Delta_i = \Delta$.**
  - **This can be done, in principle, for any dimensionality D…**



…but the required number of bins grows exponen-tially with D!

Image source: C.M. Bishop, 2006

Histograms

- The bin width Δ acts as a smoothing factor.

not smooth enough

about OK

too smooth

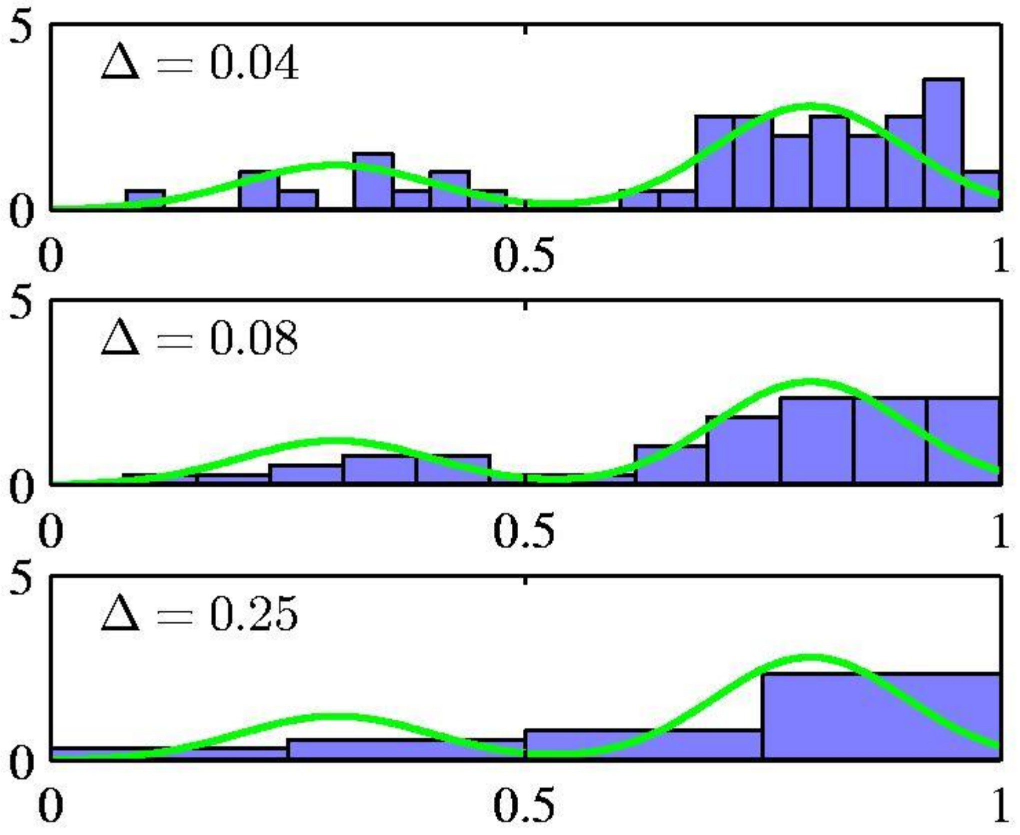


Image source: C.M. Bishop, 2006

Summary: Histograms

- Properties
  - Very general. In the limit ($N \to \infty$), every probability density can be represented.
  - No need to store the data points once histogram is computed.
  - Rather brute-force
- Problems
  - High-dimensional feature spaces
    - D-dimensional space with M bins/dimension will require M^D bins!

    $\Rightarrow$ Requires an exponentially growing number of data points

    $\Rightarrow$ "Curse of dimensionality"
  - Discontinuities at bin edges
  - Bin size?
    - too large: too much smoothing
    - too small: too much noise

Statistically Better-Founded Approach

- Data point $\mathbf{x}$ comes from pdf $p(\mathbf{x})$
  - Probability that x falls into small region $\mathcal{R}$

$$P = \int_{\mathcal{R}} p(y) dy$$

- If $\mathcal{R}$ is sufficiently small, $p(\mathbf{x})$ is roughly constant
  - Let V be the volume of $\mathcal{R}$

$$P = \int_{\mathcal{R}} p(y) dy \approx p(\mathbf{x}) V$$

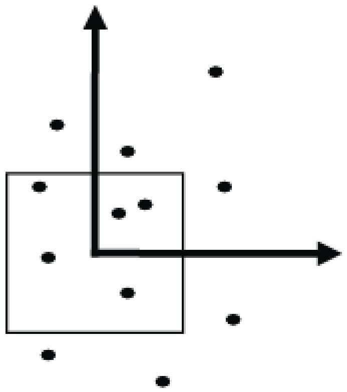
- If the number N of samples is sufficiently large, we can estimate P as

$$P = \frac{K}{N} \qquad \Rightarrow p(\mathbf{x}) \approx \frac{K}{NV}$$

Statistically Better-Founded Approach

$$\begin{tikzcd} & p(\mathbf{x}) \approx \dfrac{K}{NV} \arrow[dl, "\begin{array}{c}\text{fixed } V \\ \text{determine } K\end{array}"'] \arrow[dr, "\begin{array}{c}\text{fixed } K \\ \text{determine } V\end{array}"] & \\ \text{Kernel Methods} & & \text{K-Nearest Neighbor} \end{tikzcd}$$

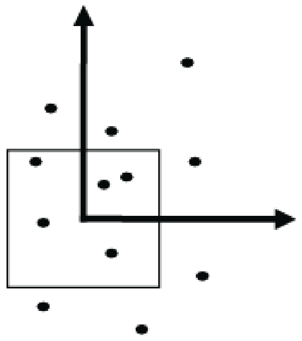
- Kernel methods
  - Example: Determine the number K of data points inside a fixed window…



Kernel Methods

- **Parzen Window**
  - **Hypercube of dimension D with edge length h:**

$$k(\mathbf{u}) = \begin{cases} 1, & |u_i \cdot \frac{1}{2}, \quad i = 1, \ldots, D \\ 0, & else \end{cases}$$



"Kernel function"

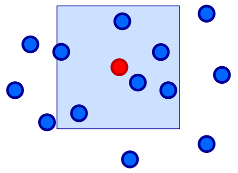
$$K = \sum_{n=1}^{N} k(\frac{\mathbf{x} - \mathbf{x}_n}{h}) \qquad V = \int k(\mathbf{u}) d\mathbf{u} = h^d$$

  - **Probability density estimate:**

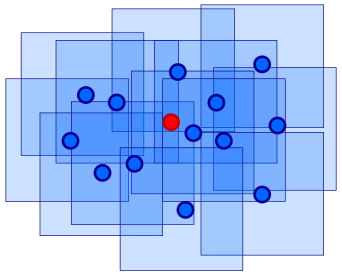
$$p(\mathbf{x}) \approx \frac{K}{NV} = \frac{1}{Nh^D} \sum_{n=1}^{N} k(\frac{\mathbf{x} - \mathbf{x}_n}{h})$$

Kernel Methods: Parzen Window

- Interpretations
  1. We place a *kernel window* k *at location* $\mathbf{x}$ and count how many data points fall inside it.



  2. We place a *kernel window* k *around each data point* $\mathbf{x}_n$ and sum up their influences at location $\mathbf{x}$.



  $\Rightarrow$ Direct visualization of the density.

- Still, we have artificial discontinuities at the cube boundaries…
  - We can obtain a smoother density model if we choose a smoother kernel function, e.g. a Gaussian

Kernel Methods: Gaussian Kernel

- Gaussian kernel
  - Kernel function

$$k(\mathbf{u}) = \frac{1}{(2\pi h^2)^{1/2}} \exp\left\{-\frac{\mathbf{u}^2}{2h^2}\right\}$$

$$K = \sum_{n=1}^{N} k(\mathbf{x} - \mathbf{x}_n) \qquad V = \int k(\mathbf{u}) d\mathbf{u} = 1$$

  - Probability density estimate

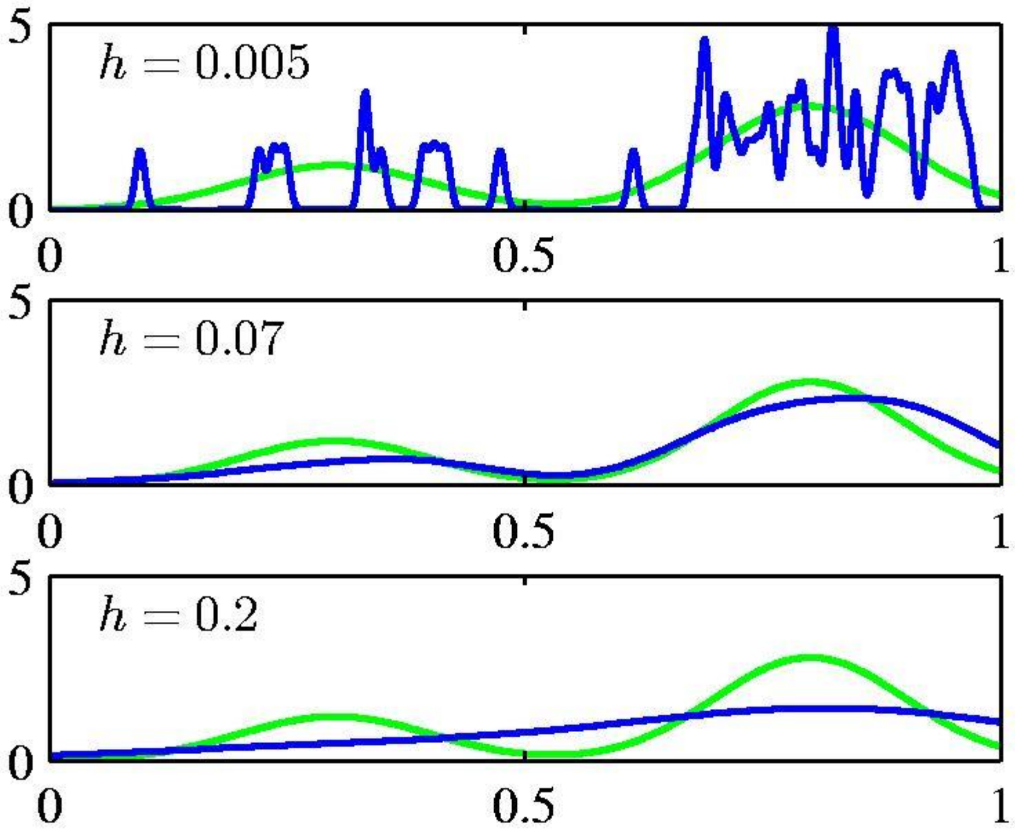
$$p(\mathbf{x}) \approx \frac{K}{NV} = \frac{1}{N} \sum_{n=1}^{N} \frac{1}{(2\pi)^{D/2} h} \exp\left\{-\frac{||\mathbf{x} - \mathbf{x}_n||^2}{2h^2}\right\}$$

Gauss Kernel: Examples

not smooth enough

about OK

too smooth



h **acts as a smoother.**

Kernel Methods

- In general
  - Any kernel such that

$$k(\mathbf{u}) \geqslant 0, \qquad \int k(\mathbf{u})\,\mathrm{d}\mathbf{u} = 1$$

can be used. Then

$$K = \sum_{n=1}^{N} k(\mathbf{x} - \mathbf{x}_n)$$

  - And we get the probability density estimate

$$p(\mathbf{x}) \approx \frac{K}{NV} = \frac{1}{N}\sum_{n=1}^{N} k(\mathbf{x} - \mathbf{x}_n)$$

Statistically Better-Founded Approach

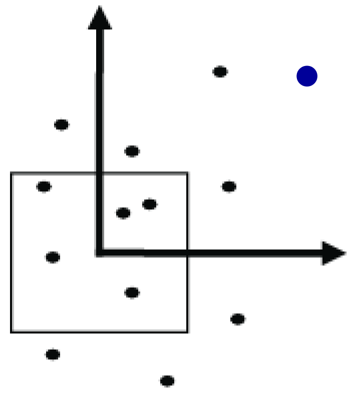
$$p(\mathbf{x}) \approx \frac{K}{NV}$$

fixed V
determine K

fixed K
determine V

Kernel Methods

K-Nearest Neighbor

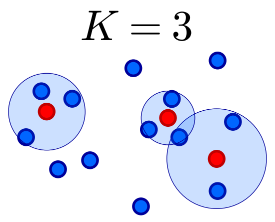


- **K-Nearest Neighbor**
  - **Increase the volume V until the K next data points are found.**

K-Nearest Neighbor

- Nearest-Neighbor density estimation
  - Fix K, estimate V from the data.
  - Consider a hypersphere centred on $\mathbf{x}$ and let it grow to a volume $V^\star$ that includes K of the given N data points.
  - Then

$$p(\mathbf{x}) \simeq \frac{K}{NV^\star}.$$



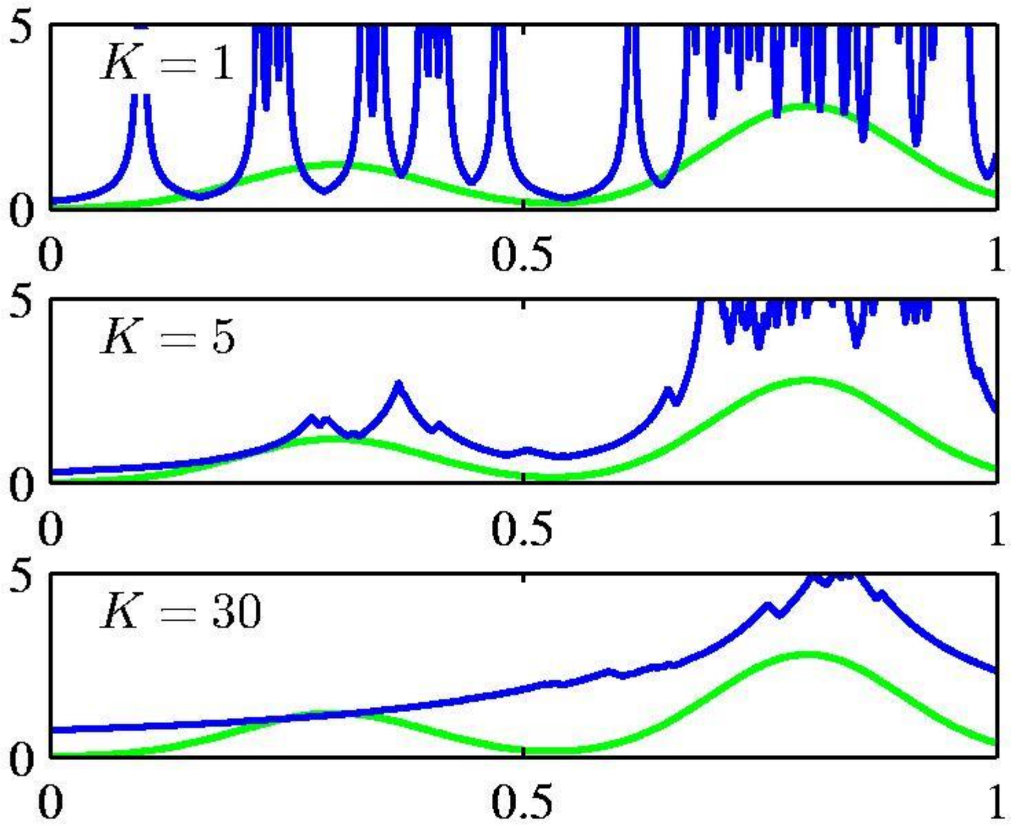
- Side note
  - Strictly speaking, the model produced by K-NN is not a true density model, because the integral over all space diverges.
  - E.g. consider $K = 1$ and a sample exactly on a data point $\mathbf{x} = x_j$.

k-Nearest Neighbor: Examples

not smooth enough

about OK

too smooth



K **acts as a smoother.**

Image source: C.M. Bishop, 2006

Summary: Kernel and k-NN Density Estimation

- Properties
  - Very general. In the limit ($N \rightarrow \infty$), every probability density can be represented.
  - No computation involved in the training phase

  $\Rightarrow$ Simply storage of the training set
- Problems
  - Requires storing and computing with the entire dataset.

  $\Rightarrow$ Computational cost linear in the number of data points.

  $\Rightarrow$ This can be improved, at the expense of some computation during training, by constructing efficient tree-based search structures.
  - Kernel size / K in K-NN?
    - Too large: too much smoothing
    - Too small: too much noise

K-Nearest Neighbor Classification

- Bayesian Classification

$$p(\mathcal{C}_j|\mathbf{x}) = \frac{p(\mathbf{x}|\mathcal{C}_j)p(\mathcal{C}_j)}{p(\mathbf{x})}$$

- Here we have

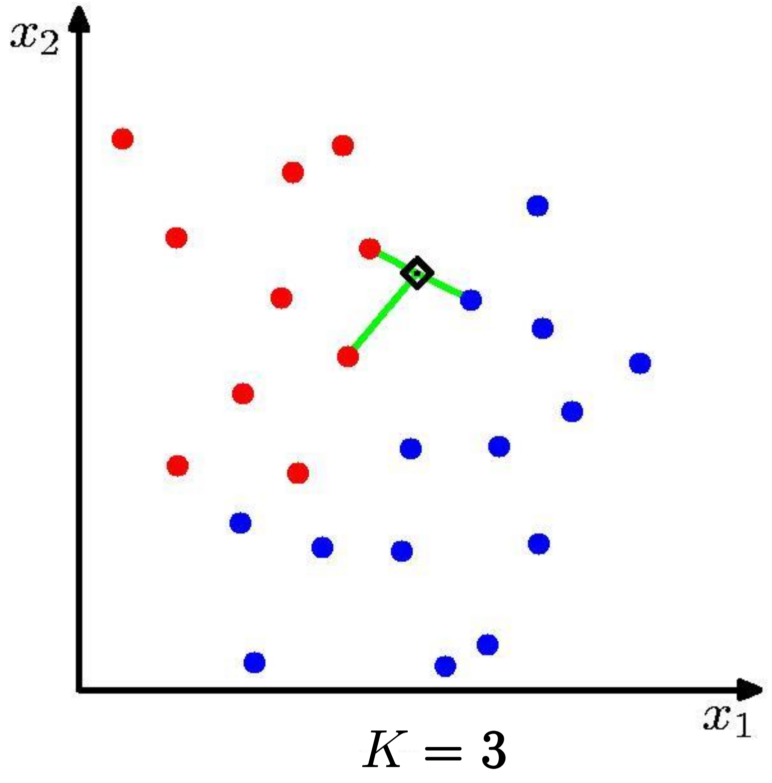
$$p(\mathbf{x}) \approx \frac{K}{NV}$$

$$p(\mathbf{x}|\mathcal{C}_j) \approx \frac{K_j}{N_j V} \longrightarrow p(\mathcal{C}_j|\mathbf{x}) \approx \frac{K_j}{N_j V}\frac{N_j}{N}\frac{NV}{K} = \frac{K_j}{K}$$

$$p(\mathcal{C}_j) \approx \frac{N_j}{N}$$

k-Nearest Neighbor classification

K-Nearest Neighbors for Classification



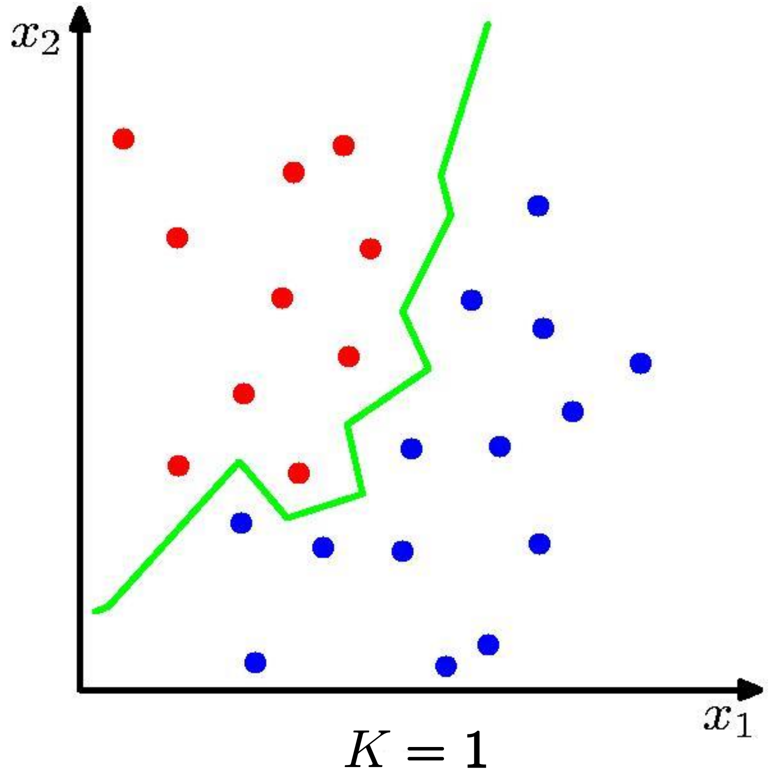
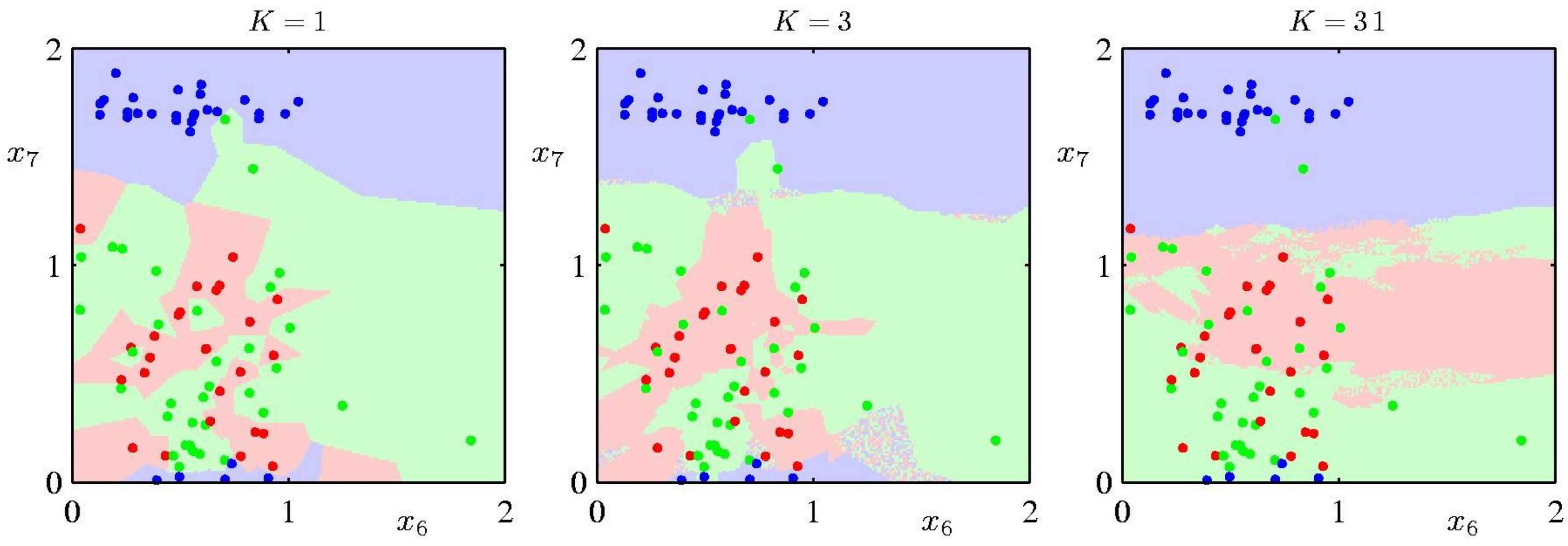


Image source: C.M. Bishop, 2006

K-Nearest Neighbors for Classification

- Results on an example data set



- K acts as a smoothing parameter.
- Theoretical guarantee
  - For $N \to \infty$, the error rate of the 1-NN classifier is never more than twice the optimal error (obtained from the true conditional class distributions).

Image source: C.M. Bishop, 2006

Bias-Variance Tradeoff

- Probability density estimation
  - Histograms: bin size?
    - Δ too large: too smooth
    - Δ too small: not smooth enough
  - Kernel methods: kernel size?
    - h too large: too smooth
    - h too small: not smooth enough
  - K-Nearest Neighbor: K?
    - K too large: too smooth
    - K too small: not smooth enough

Too much bias

Too much variance

- This is a general problem of many probability density estimation methods
  - Including parametric methods and mixture models

Slide credit: Bernt Schiele

Discussion

- The methods discussed so far are all simple and easy to apply. They are used in many practical applications.
- However…
  - Histograms scale poorly with increasing dimensionality.

  $\Rightarrow$ Only suitable for relatively low-dimensional data.

  - Both k-NN and kernel density estimation require the entire data set to be stored.

  $\Rightarrow$ Too expensive if the data set is large.

  - Simple parametric models are very restricted in what forms of distributions they can represent.

  $\Rightarrow$ Only suitable if the data has the same general form.

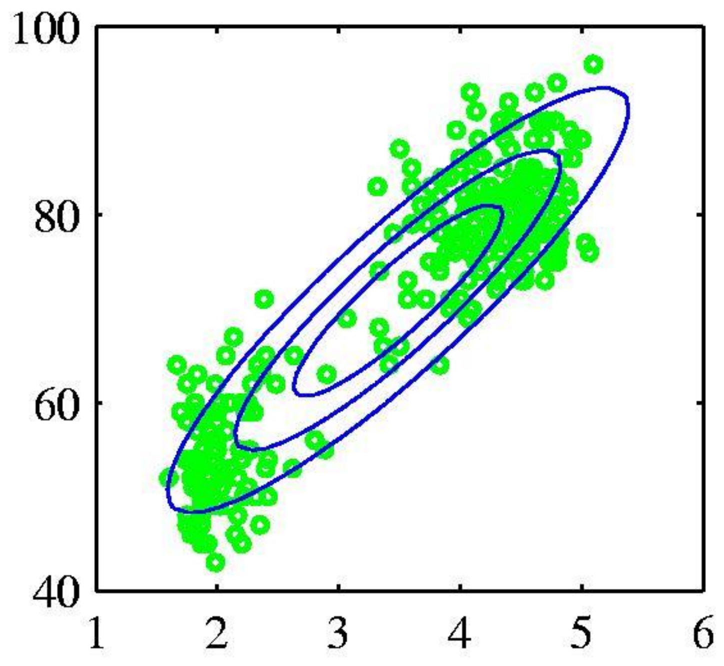
- We need density models that are efficient and flexible!

Topics of This Lecture

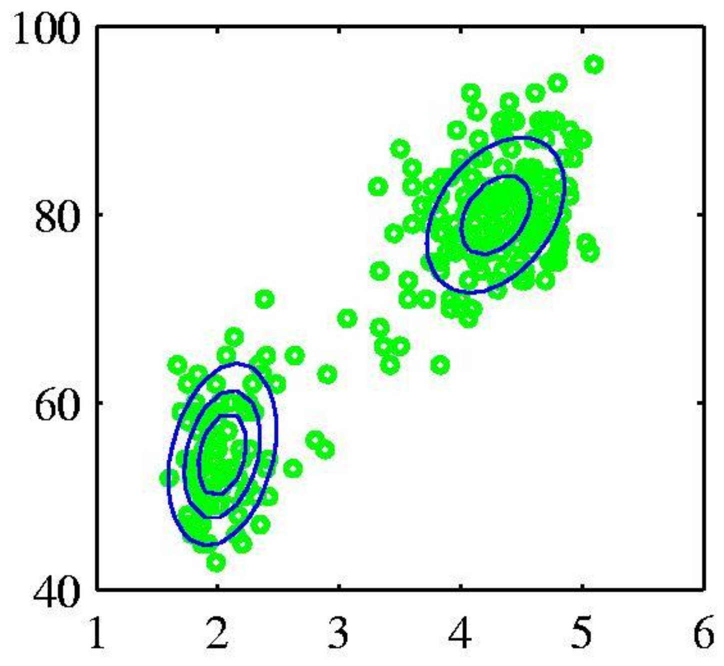
- Recap: Bayes Decision Theory
- Parametric Methods
  - Recap: Maximum Likelihood approach
  - Bayesian Learning
- Non-Parametric Methods
  - Histograms
  - Kernel density estimation
  - K-Nearest Neighbors
  - k-NN for Classification
  - Bias-Variance tradeoff
- **Mixture distributions**
  - **Mixture of Gaussians (MoG)**
  - **Maximum Likelihood estimation attempt**

Mixture Distributions

- A single parametric distribution is often not sufficient
  - E.g. for multimodal data



Single Gaussian

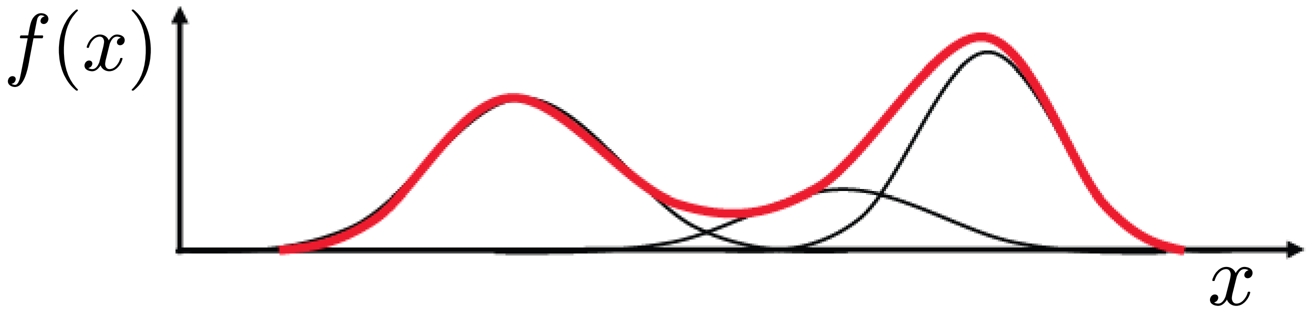


Mixture of two Gaussians

Image source: C.M. Bishop, 2006

Mixture of Gaussians (MoG)

- Sum of M individual Normal distributions



  - In the limit, every smooth distribution can be approximated this way (if M is large enough)

$$p(x|\theta) = \sum_{j=1}^{M} p(x|\theta_j)p(j)$$

Mixture of Gaussians

$$p(x|\theta) = \sum_{j=1}^{M} \boxed{p(x|\theta_j)}\boxed{p(j)}$$

Likelihood of measurement x given mixture component j

$$p(x|\theta_j) = \mathcal{N}(x|\mu_j, \sigma_j^2) = \frac{1}{\sqrt{2\pi}\sigma_j} \exp\left\{ -\frac{(x-\mu_j)^2}{2\sigma_j^2} \right\}$$

$$p(j) = \pi_j \quad \text{with} \quad 0 \cdot \pi_j \cdot 1 \quad \text{and} \quad \sum_{j=1}^{M} \pi_j = 1.$$

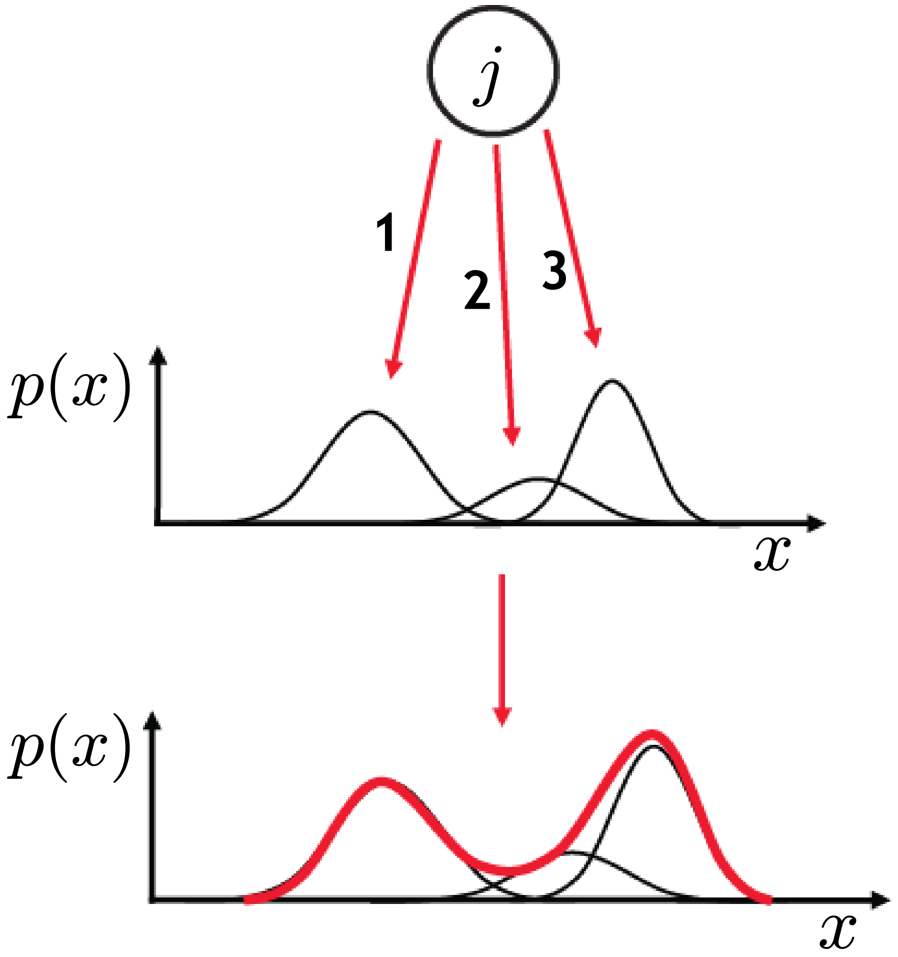
Prior of component j

- **Notes**
  - The mixture density integrates to 1: $\int p(x)dx = 1$
  - The mixture parameters are

$$\theta = (\pi_1, \mu_1, \sigma_1, \ldots, , \pi_M, \mu_M, \sigma_M)$$

Mixture of Gaussians (MoG)

- "Generative model"



$p(j) = \pi_j$ "Weight" of mixture component

$p(x|\theta_j)$ Mixture component

Mixture density

$$p(x|\theta) = \sum_{j=1}^{M} p(x|\theta_j) p(j)$$

Mixture of Multivariate Gaussians

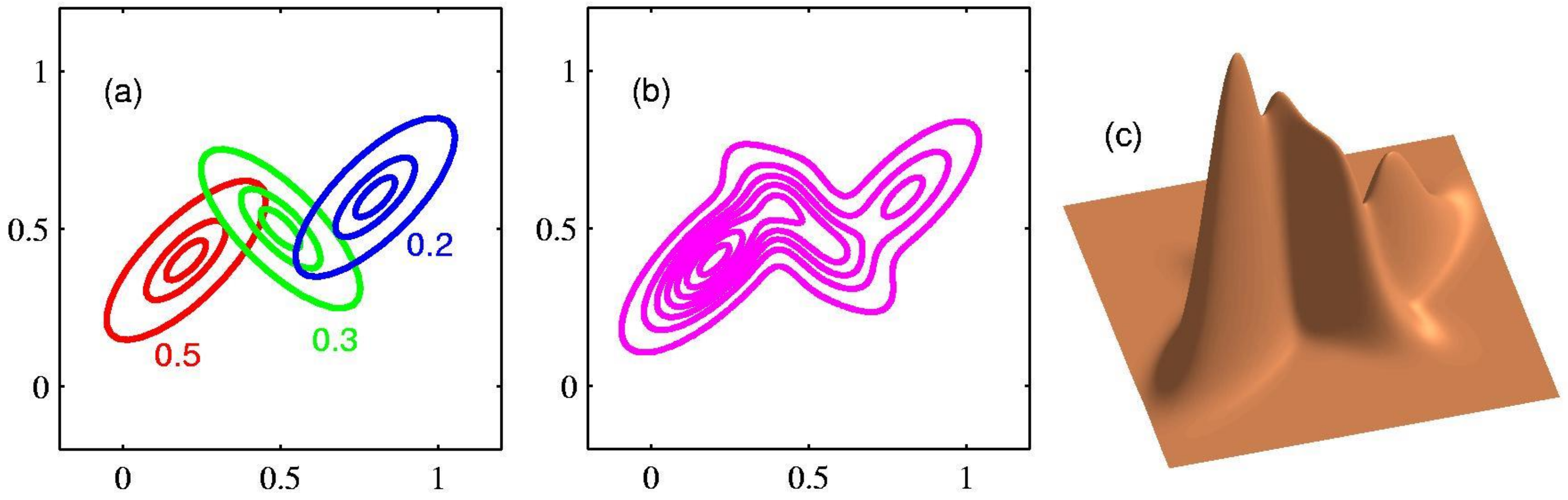


Image source: C.M. Bishop, 2006

Mixture of Multivariate Gaussians

- **Multivariate Gaussians**

$$p(\mathbf{x}|\theta) = \sum_{j=1}^{M} p(\mathbf{x}|\theta_j)p(j)$$

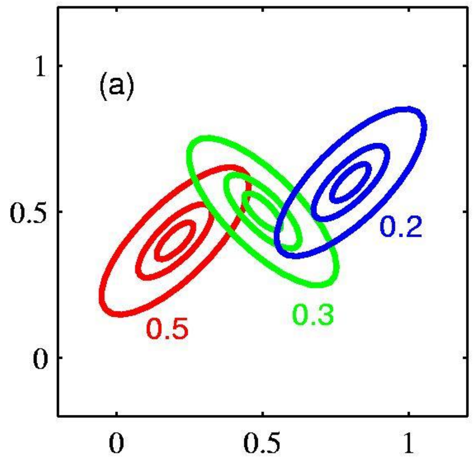
$$p(\mathbf{x}|\theta_j) = \frac{1}{(2\pi)^{D/2}|\mathbf{\Sigma}_j|^{1/2}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}_j)^{\mathrm{T}}\mathbf{\Sigma}_j^{-1}(\mathbf{x}-\boldsymbol{\mu}_j)\right\}$$

  - **Mixture weights / mixture coefficients:**

$p(j) = \pi_j$ **with** $0 \cdot \ \pi_j \cdot \ 1$ **and** $\sum_{j=1}^{M} \pi_j = 1$

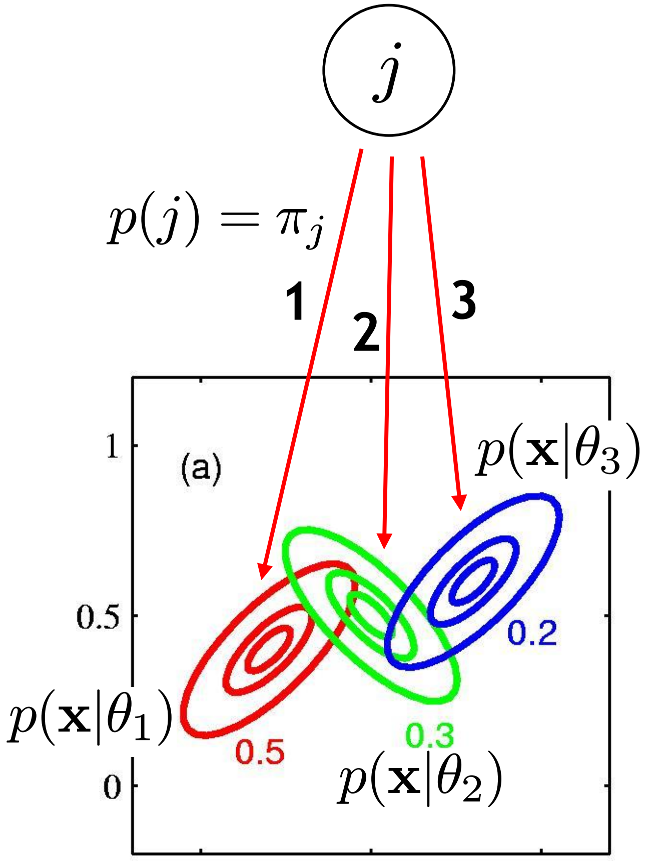
  - **Parameters:**

$$\theta = (\pi_1, \boldsymbol{\mu}_1, \mathbf{\Sigma}_1, \ldots, \pi_M, \boldsymbol{\mu}_M, \mathbf{\Sigma}_M)$$

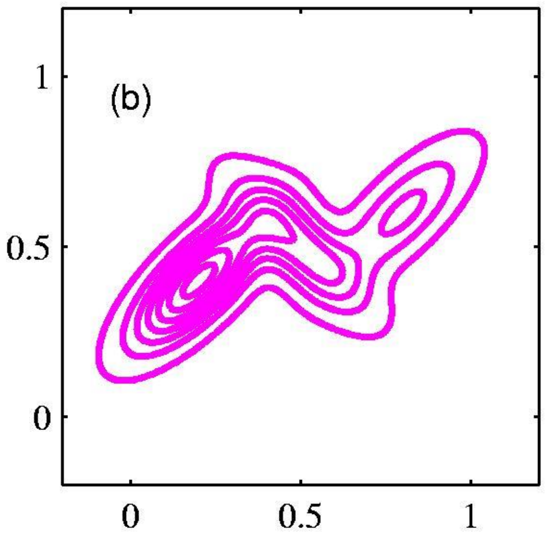


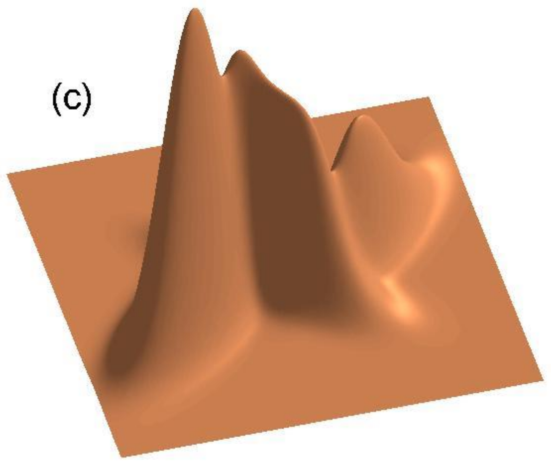
Mixture of Multivariate Gaussians

- "Generative model"



$$p(\mathbf{x}|\theta) = \sum_{j=1}^{3} \pi_j p(\mathbf{x}|\theta_j)$$

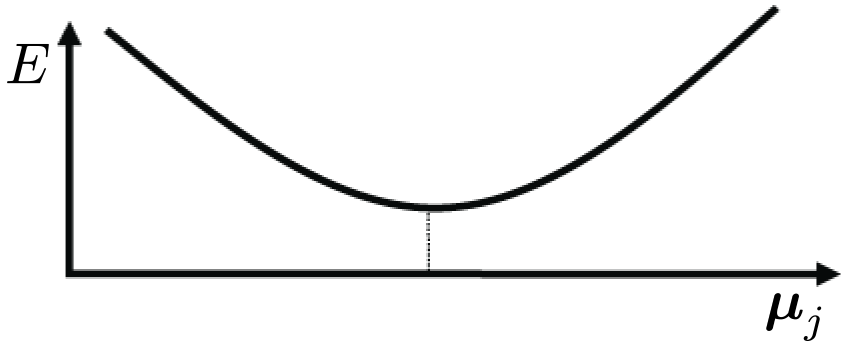




Mixture of Gaussians - 1st Estimation Attempt

- **Maximum Likelihood**
  - **Minimize** $E = -\ln L(\theta) = -\sum_{n=1}^{N} \ln p(\mathbf{x}_n|\theta)$
  - **Let's first look at μ_j:**

$$\frac{\partial E}{\partial \boldsymbol{\mu}_j} = 0$$



  - **We can already see that this will be difficult, since**

$$\ln p(\mathbf{X}|\boldsymbol{\pi}, \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \sum_{n=1}^{N} \ln \left\{ \sum_{k=1}^{K} \pi_k \mathcal{N}(\mathbf{x}_n|\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k) \right\}$$

This will cause problems!

Mixture of Gaussians – 1st Estimation Attempt

- **Minimization:**

$$\boxed{\frac{\partial}{\partial \boldsymbol{\mu}_j}\mathcal{N}(\mathbf{x}_n|\boldsymbol{\mu}_k,\boldsymbol{\Sigma}_k) = \boldsymbol{\Sigma}^{-1}(\mathbf{x}_n-\boldsymbol{\mu}_j)\mathcal{N}(\mathbf{x}_n|\boldsymbol{\mu}_k,\boldsymbol{\Sigma}_k)}$$

$$\frac{\partial E}{\partial \boldsymbol{\mu}_j} = -\sum_{n=1}^{N} \frac{\frac{\partial}{\partial \boldsymbol{\mu}_j} p(\mathbf{x}_n|\theta_j)}{\sum_{k=1}^{K} p(\mathbf{x}_n|\theta_k)}$$

$$= -\sum_{n=1}^{N} \left(\boldsymbol{\Sigma}^{-1}(\mathbf{x}_n-\boldsymbol{\mu}_j) \frac{p(\mathbf{x}_n|\theta_j)}{\sum_{k=1}^{K} p(\mathbf{x}_n|\theta_k)} \right)$$

$$= -\cancel{\boldsymbol{\Sigma}}^{-1} \sum_{n=1}^{N} (\mathbf{x}_n-\boldsymbol{\mu}_j) \underbrace{\frac{\pi_j \mathcal{N}(\mathbf{x}_n|\boldsymbol{\mu}_j,\boldsymbol{\Sigma}_j)}{\sum_{k=1}^{K} \pi_k \mathcal{N}(\mathbf{x}_n|\boldsymbol{\mu}_k,\boldsymbol{\Sigma}_k)}}_{=\gamma_j(\mathbf{x}_n)} \overset{!}{=} 0$$

"responsibility" of component j for $\mathbf{x}_n$

- **We thus obtain**

$$\Rightarrow \boldsymbol{\mu}_j = \frac{\sum_{n=1}^{N} \gamma_j(\mathbf{x}_n)\mathbf{x}_n}{\sum_{n=1}^{N} \gamma_j(\mathbf{x}_n)}$$

Mixture of Gaussians – 1st Estimation Attempt

- But…

$$\boldsymbol{\mu}_j = \frac{\sum_{n=1}^{N} \gamma_j(\mathbf{x}_n)\mathbf{x}_n}{\sum_{n=1}^{N} \gamma_j(\mathbf{x}_n)} \qquad \gamma_j(\mathbf{x}_n) = \frac{\pi_j \mathcal{N}(\mathbf{x}_n|\boldsymbol{\mu}_j, \boldsymbol{\Sigma}_j)}{\sum_{k=1}^{K} \pi_k \mathcal{N}(\mathbf{x}_n|\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)}$$

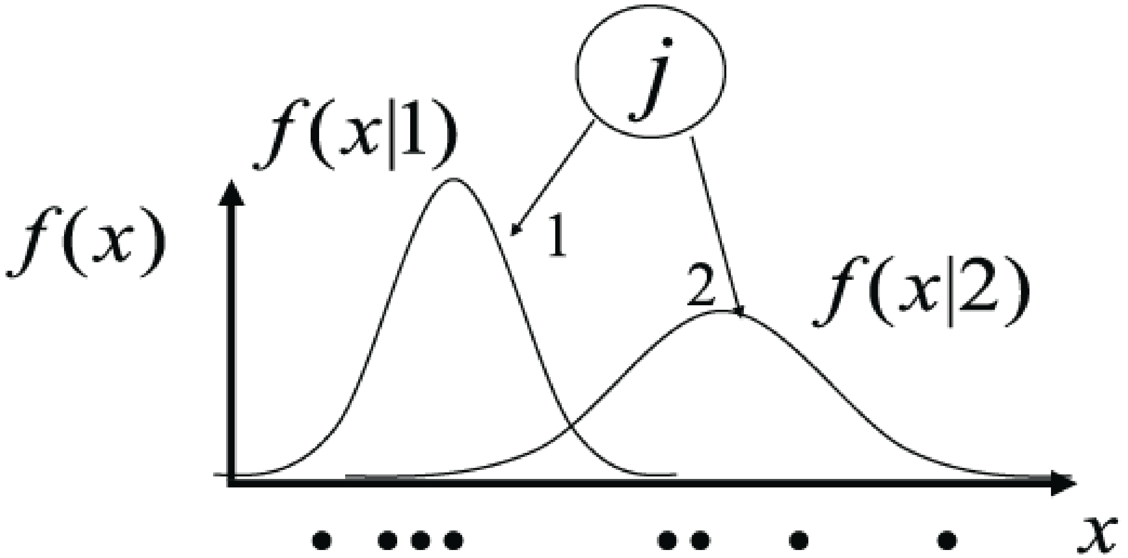
- I.e. there is no direct analytical solution!

$$\frac{\partial E}{\partial \boldsymbol{\mu}_j} = f\left(\pi_1, \boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1, \ldots, \pi_M, \boldsymbol{\mu}_M, \boldsymbol{\Sigma}_M\right)$$

  - Complex gradient function (non-linear mutual dependencies)
  - Optimization of one Gaussian depends on all other Gaussians!
  - It is possible to apply iterative numerical optimization here, but in the following, we will see a simpler method.

Mixture of Gaussians – Other Strategy

- **Other strategy:**



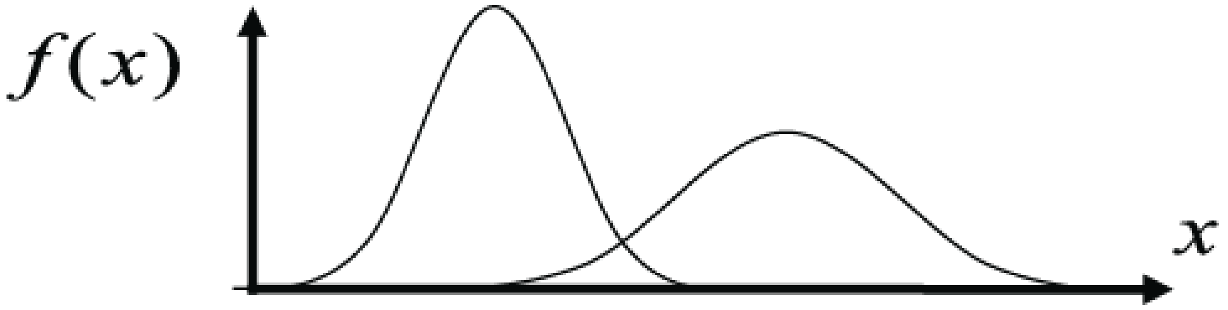
  - **Observed data:** ● ●●● ●● ● ●
  - **Unobserved data:** 1 111 22 2 2
    - Unobserved = "hidden variable": j|x

$$h(j=1|x_n) = \quad 1 \;\; 111 \qquad 00 \quad 0 \quad 0$$

$$h(j=2|x_n) = \quad 0 \;\; 000 \qquad 11 \quad 1 \quad 1$$

Mixture of Gaussians – Other Strategy

- Assuming we knew the values of the hidden variable…



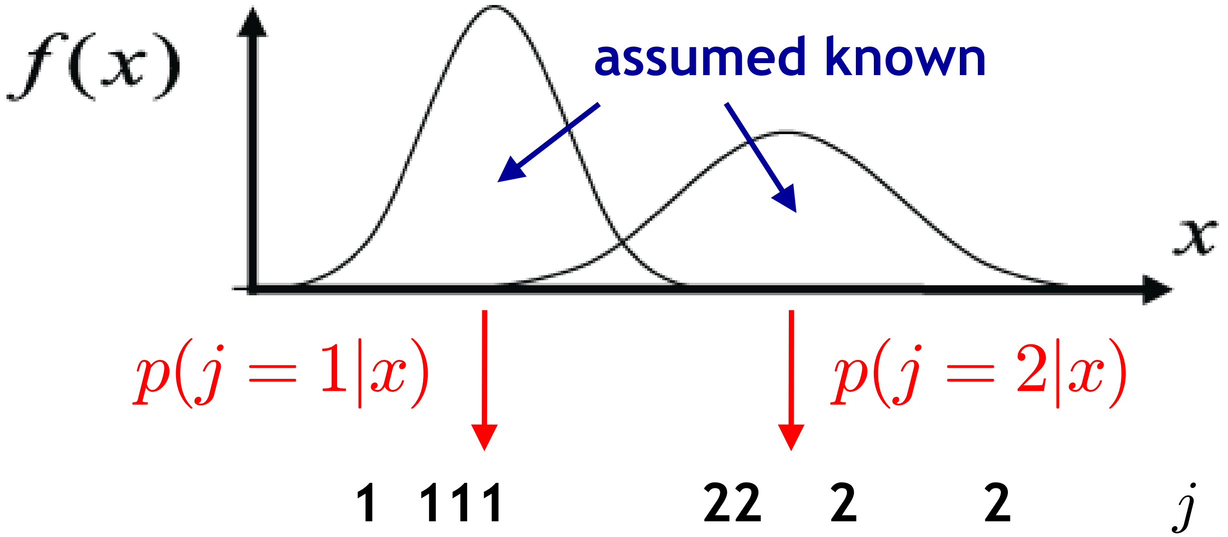
ML for Gaussian #1 ML for Gaussian #2

assumed known $\longrightarrow$	1	111	22	2	2	j
$h(j=1\|x_n) =$	1	111	00	0	0	
$h(j=2\|x_n) =$	0	000	11	1	1	

$$\mu_1 = \frac{\sum_{n=1}^{N} h(j=1|x_n)x_n}{\sum_{i=1}^{N} h(j=1|x_n)} \qquad \mu_2 = \frac{\sum_{n=1}^{N} h(j=2|x_n)x_n}{\sum_{i=1}^{N} h(j=2|x_n)}$$

Mixture of Gaussians – Other Strategy

- Assuming we knew the mixture components...

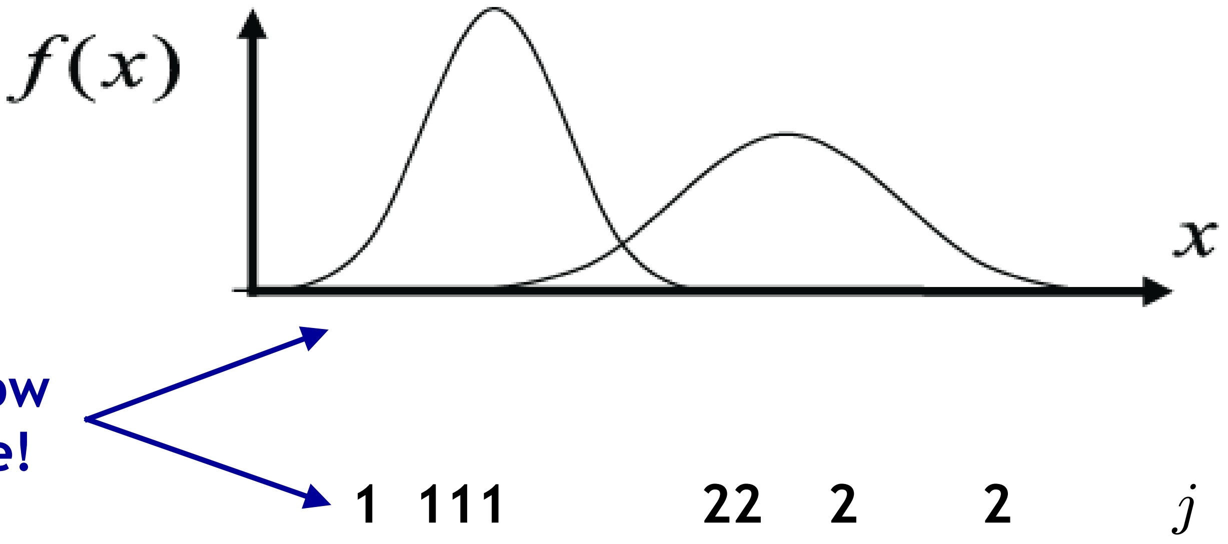


- Bayes decision rule: Decide $j = 1$ if

$$p(j = 1|x_n) > p(j = 2|x_n)$$

Mixture of Gaussians – Other Strategy

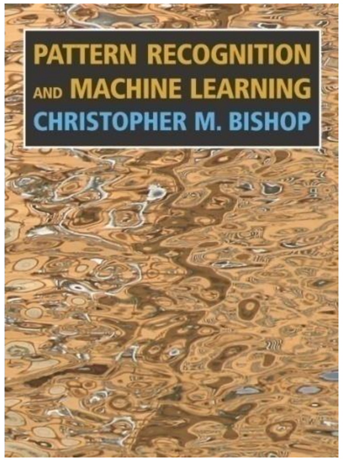
- Chicken and egg problem – what comes first?



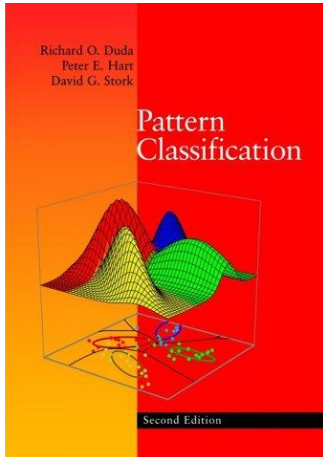
- In order to break the loop, we need an estimate for j.
  - E.g. by clustering…
  - $\Rightarrow$ Next lecture…

References and Further Reading

- More information in Bishop's book
  - Gaussian distribution and ML: Ch. 1.2.4 and 2.3.1-2.3.4.
  - Bayesian Learning: Ch. 1.2.3 and 2.3.6.
  - Nonparametric methods: Ch. 2.5.
- Additional information can be found in Duda & Hart
  - ML estimation: Ch. 3.2
  - Bayesian Learning: Ch. 3.3-3.5
  - Nonparametric methods: Ch. 4.1-4.5



Christopher M. Bishop
Pattern Recognition and Machine Learning
Springer, 2006



R.O. Duda, P.E. Hart, D.G. Stork
Pattern Classification
2nd Ed., Wiley-Interscience, 2000