Machine Learning – Lecture 1

Introduction

09.04.2015

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Many slides adapted from B. Schiele

Organization

- Lecturer
  - Prof. Bastian Leibe (leibe@vision.rwth-aachen.de)
- Assistants
  - Ishrat Badami (badami@vision.rwth-aachen.de)
  - Michael Kramp (kramp@vision.rwth-aachen.de)
- Course webpage
  - http://www.vision.rwth-aachen.de/teaching/
  - Slides will be made available on the webpage
  - There is also an L2P electronic repository
- Please subscribe to the lecture on the Campus system!
  - Important to get email announcements and L2P access!

Language

- Official course language will be English
  - If at least one English-speaking student is present.
  - If not… you can choose.
- However…
  - Please tell me when I'm talking too fast or when I should repeat something in German for better understanding!
  - You may at any time ask questions in German!
  - You may turn in your exercises in German.
  - You may take the oral exam in German.

Organization

- Structure: 3V (lecture) + 1Ü (exercises)
  - 6 EECS credits
  - Part of the area "Applied Computer Science"
- Place & Time
  - Lecture: Tue 14:15 - 15:45 room UMIC 025
  - Lecture/Exercises: Thu 14:15 - 15:45 room UMIC 025
- Exam
  - Written exam
  - Towards the end of the semester, there will be a proposed date

Exercises and Supplementary Material

- Exercises
  - Typically 1 exercise sheet every 2 weeks.
  - Pen & paper and Matlab based exercises
  - Hands-on experience with the algorithms from the lecture.
  - Send your solutions the night before the exercise class.
  - Need to reach ≥ 50% of the points to qualify for the exam!
- Teams are encouraged!
  - You can form teams of up to 3 people for the exercises.
  - Each team should only turn in one solution.
  - But list the names of all team members in the submission.

Course Webpage

Tentative Schedule

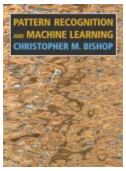
Date	Topic	Content	Slides	Related Material
07.04.15	*no class*	-	-	-
09.04.15	**Introduction**	Introduction, Probability Theory, Bayes Decision Theory, Minimizing Expected Loss	pdf, fullpage	Bishop Ch. 1.1, 1.2.1-1.2.3, 1.5.1-1.5.4
14.04.15	*Exercise 0*	*Intro Matlab*		-
16.04.15	**Prob. Density Estimation I**	Nonparametric Methods, Histograms, Kernel Density Estimation, Parametric Methods, Gaussian Distribution, Maximum Likelihood, Bayesian Learning, Bias-Variance Problem		Bishop Ch. 2.5, 1.2.4, 2.3.1-2.3.4
21.04.15	**Prob. Density Estimation II**	Mixture of Gaussians, k-Means Clustering, EM-Clustering, EM Algorithm		Bishop chapter 9, original Dempster&Laird EM paper, Bilmes' EM tutorial
23.04.15	**Linear Discriminant Functions**	Linear Discriminant Functions, Least-squares Classification, Generalized Linear Models		Bishop chapter 4.1

Exercise on Tuesday

http://www.vision.rwth-aachen.de/teaching/

Textbooks

- Most lecture topics will be covered in Bishop's book.
- Some additional topics can be found in Duda & Hart.



Christopher M. Bishop
Pattern Recognition and Machine Learning
Springer, 2006

(available in the library's "Handapparat")

R.O. Duda, P.E. Hart, D.G. Stork
Pattern Classification
2^{nd} Ed., Wiley-Interscience, 2000

- Research papers will be given out for some topics.
  - Tutorials and deeper introductions.
  - Application papers

How to Find Us

- Office:
  - UMIC Research Centre
  - Mies-van-der-Rohe-Strasse 15, room 124

- Office hours
  - If you have questions to the lecture, come to Ishrat or Michael.
  - My regular office hours will be announced (additional slots are available upon request)
  - Send us an email before to confirm a time slot.

Questions are welcome!

Machine Learning

- Statistical Machine Learning
  - Principles, methods, and algorithms for learning and prediction on the basis of past evidence
- Already everywhere
  - Speech recognition (e.g. speed-dialing)
  - Computer vision (e.g. face detection)
  - Hand-written character recognition (e.g. letter delivery)
  - Information retrieval (e.g. image & video indexing)
  - Operation systems (e.g. caching)
  - Fraud detection (e.g. credit cards)
  - Text filtering (e.g. email spam filters)
  - Game playing (e.g. strategy prediction)
  - Robotics (e.g. prediction of battery lifetime)

Slide credit: Bernt Schiele

Machine Learning

- Goal
  - *Machines that learn to perform a task from experience*
- Why?
  - Crucial component of every intelligent/autonomous system
  - Important for a system's adaptability
  - Important for a system's generalization capabilities
  - Attempt to understand human learning

Slide credit: Bernt Schiele

Machine Learning: Core Questions

- *Learning to perform a task from experience*
- Learning
  - Most important part here!
  - We do not want to encode the knowledge ourselves.
  - The machine should learn the relevant criteria automatically from past observations and adapt to the given situation.
- Tools
  - Statistics
  - Probability theory
  - Decision theory
  - Information theory
  - Optimization theory

Slide credit: Bernt Schiele

Machine Learning: Core Questions

- *Learning to perform a task from experience*
- Task
  - Can often be expressed through a mathematical function

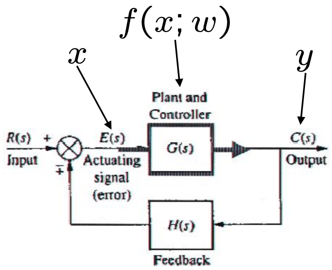
$$y = f(x; w)$$

  - x: Input
  - y: Output
  - w: Parameters (this is what is "learned")
- Classification vs. Regression
  - Regression: continuous y
  - Classification: discrete y
    - E.g. class membership, sometimes also posterior probability

Slide credit: Bernt Schiele

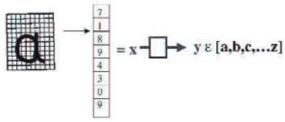
Example: Regression

- Automatic control of a vehicle

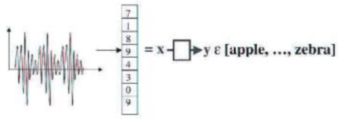


Examples: Classification

- Email filtering $x \in [\text{a-z}]^{+}$ → $y \in$ [important, spam]
- Character recognition

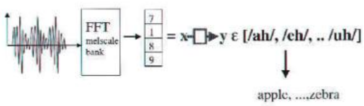


- Speech recognition



Machine Learning: Core Problems

- Input x:



- Features
  - Invariance to irrelevant input variations
  - Selecting the "right" features is crucial
  - Encoding and use of "domain knowledge"
  - Higher-dimensional features are more discriminative.
- Curse of dimensionality
  - Complexity increases exponentially with number of dimensions.

Machine Learning: Core Questions

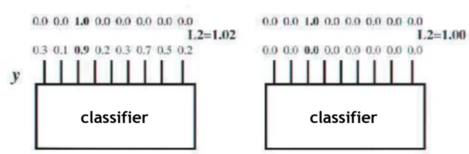
- *Learning to perform a task from experience*
- Performance: "99% correct classification"
  - Of what???
  - Characters? Words? Sentences?
  - Speaker/writer independent?
  - Over what data set?
  - …
- "The car drives without human intervention 99% of the time on country roads"

Machine Learning: Core Questions

- *Learning to perform a task from experience*
- Performance measure: Typically *one number*
  - % correctly classified letters
  - Average driving distance (until crash…)
  - % games won
  - % correctly recognized words, sentences, answers
- Generalization performance
  - Training vs. test
  - "All" data

Machine Learning: Core Questions

- *Learning to perform a task from experience*
- Performance measure: more subtle problem
  - Also necessary to compare partially correct outputs.
  - How do we weight different kinds of errors?
  - Example: L2 norm



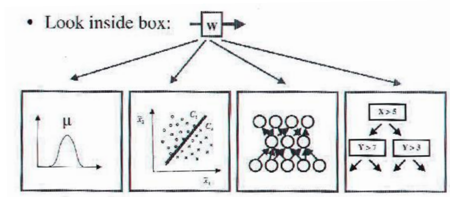
Machine Learning: Core Questions

- *Learning to perform a task from experience*

- What data is available?
  - Data with labels: *supervised learning*
    - Images / speech with target labels
    - Car sensor data with target steering signal
  - Data without labels: *unsupervised learning*
    - Automatic clustering of sounds and phonemes
    - Automatic clustering of web sites
  - Some data with, some without labels: *semi-supervised learning*
  - No examples: *learning by doing*
  - Feedback/rewards: *reinforcement learning*

Machine Learning: Core Questions

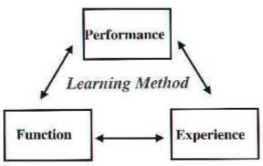
- $y = f(x; w)$
  - w: characterizes the family of functions
  - w: indexes the space of hypotheses
  - w: vector, connection matrix, graph, …



Machine Learning: Core Questions

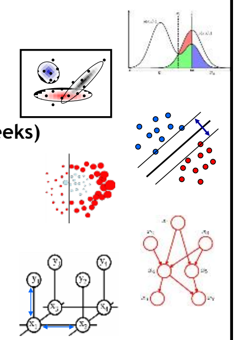
- *Learning to perform a task from experience*

- Learning
  - Most often learning = optimization
  - Search in hypothesis space
  - Search for the "best" function / model parameter w
    - I.e. maximize $y = f(x; w)$ w.r.t. the performance measure



Course Outline

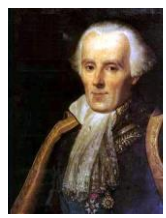
- Fundamentals (2 weeks)
  - Bayes Decision Theory
  - Probability Density Estimation
- Discriminative Approaches (5 weeks)
  - Linear Discriminant Functions
  - Support Vector Machines
  - Ensemble Methods & Boosting
  - Randomized Trees, Forests & Ferns
- Generative Models (4 weeks)
  - Bayesian Networks
  - Markov Random Fields
  - Probabilistic Inference



Topics of This Lecture

- (Re-)view: Probability Theory
  - Probabilities
  - Probability densities
  - Expectations and covariances
- Bayes Decision Theory
  - Basic concepts
  - Minimizing the misclassification rate
  - Minimizing the expected loss
  - Discriminant functions

Probability Theory



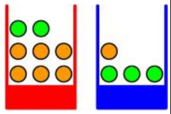
"Probability theory is nothing but common sense reduced to calculation."

Pierre-Simon de Laplace, 1749-1827

Image source: Wikipedia

Probability Theory

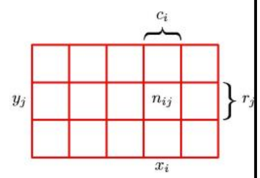
- Example: apples and oranges
  - We have two boxes to pick from.
  - Each box contains both types of fruit.
  - What is the probability of picking an apple?



- Formalization
  - Let $B \in \{r, b\}$ be a random variable for the box we pick.
  - Let $F \in \{a, o\}$ be a random variable for the type of fruit we get.
  - Suppose we pick the red box 40% of the time. We write this as
    $$p(B=r)=0.4 \qquad p(B=b)=0.6$$
  - The probability of picking an apple *given* a choice for the box is
    $$p(F=a \mid B=r)=0.25 \qquad p(F=a \mid B=b)=0.75$$
  - What is the probability of picking an apple?
    $$p(F=a)=?$$

Image source: C.M. Bishop, 2006

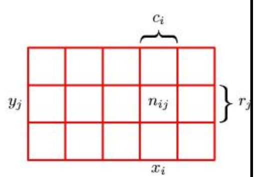
Probability Theory



- More general case
  - Consider two random variables $X \in \{x_i\}$ and $Y \in \{y_j\}$
  - Consider N trials and let
    $$\begin{aligned} n_{ij} &= \#\{X = x_i \wedge Y = y_j\} \\ c_i &= \#\{X = x_i\} \\ r_j &= \#\{Y = y_j\} \end{aligned}$$
- Then we can derive
  - Joint probability $\quad p(X=x_i, Y=y_j) = \frac{n_{ij}}{N}$
  - Marginal probability $\quad p(X=x_i) = \frac{c_i}{N}.$
  - Conditional probability $\quad p(Y=y_j \mid X=x_i) = \frac{n_{ij}}{c_i}$

Image source: C.M. Bishop, 2006

Probability Theory



- Rules of probability
  - Sum rule
    $$p(X=x_i) = \frac{c_i}{N} = \frac{1}{N}\sum_{j=1}^{L} n_{ij} = \sum_{j=1}^{L} p(X=x_i, Y=y_j)$$
  - Product rule
    $$\begin{aligned} p(X=x_i, Y=y_j) &= \frac{n_{ij}}{N} = \frac{n_{ij}}{c_i} \cdot \frac{c_i}{N} \\ &= p(Y=y_j \mid X=x_i)\,p(X=x_i) \end{aligned}$$

Image source: C.M. Bishop, 2006

The Rules of Probability

- Thus we have

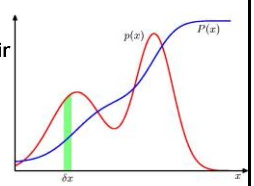
Sum Rule	$p(X) = \sum_Y p(X,Y)$
Product Rule	$p(X,Y) = p(Y \mid X)p(X)$

- From those, we can derive

Bayes' Theorem	$p(Y \mid X) = \frac{p(X \mid Y)p(Y)}{p(X)}$
where	$p(X) = \sum_Y p(X \mid Y)p(Y)$

Probability Densities

- Probabilities over continuous variables are defined over their probability density function (pdf) $p(x)$.
  $$p(x \in (a,b)) = \int_a^b p(x)\,\mathrm{d}x$$



- The probability that x lies in the interval $(-\infty, z)$ is given by the cumulative distribution function
  $$P(z) = \int_{-\infty}^{z} p(x)\,\mathrm{d}x$$

Image source: C.M. Bishop, 2006

Expectations

- The average value of some function $f(x)$ under a probability distribution $p(x)$ is called its expectation
  $$\mathbb{E}[f] = \sum_x p(x)f(x) \qquad \mathbb{E}[f] = \int p(x)f(x)\,\mathrm{d}x$$
  discrete case $\qquad$ continuous case
- If we have a finite number N of samples drawn from a pdf, then the expectation can be approximated by
  $$\mathbb{E}[f] \simeq \frac{1}{N}\sum_{n=1}^{N} f(x_n)$$
- We can also consider a conditional expectation
  $$\mathbb{E}_x[f \mid y] = \sum_x p(x \mid y) f(x)$$

Variances and Covariances

- The variance provides a measure how much variability there is in $f(x)$ around its mean value $\mathbb{E}[f(x)]$.

$$\mathrm{var}[f] = \mathbb{E}\left[(f(x) - \mathbb{E}[f(x)])^2\right] = \mathbb{E}[f(x)^2] - \mathbb{E}[f(x)]^2$$

- For two random variables x and y, the covariance is defined by

$$\begin{aligned}\mathrm{cov}[x,y] &= \mathbb{E}_{x,y}\left[\{x - \mathbb{E}[x]\}\{y - \mathbb{E}[y]\}\right] \\ &= \mathbb{E}_{x,y}[xy] - \mathbb{E}[x]\mathbb{E}[y]\end{aligned}$$

- If $\mathbf{x}$ and $\mathbf{y}$ are vectors, the result is a covariance matrix

$$\begin{aligned}\mathrm{cov}[\mathbf{x},\mathbf{y}] &= \mathbb{E}_{\mathbf{x},\mathbf{y}}\left[\{\mathbf{x} - \mathbb{E}[\mathbf{x}]\}\{\mathbf{y}^{\mathrm{T}} - \mathbb{E}[\mathbf{y}^{\mathrm{T}}]\}\right] \\ &= \mathbb{E}_{\mathbf{x},\mathbf{y}}[\mathbf{x}\mathbf{y}^{\mathrm{T}}] - \mathbb{E}[\mathbf{x}]\mathbb{E}[\mathbf{y}^{\mathrm{T}}]\end{aligned}$$

Bayes Decision Theory

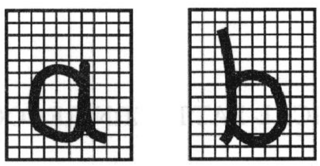
Thomas Bayes, 1701-1761

"The theory of inverse probability is founded upon an error, and must be wholly rejected."

R.A. Fisher, 1925

Bayes Decision Theory

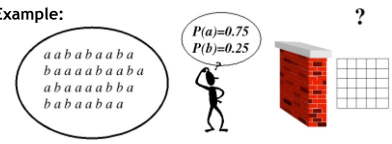
- Example: handwritten character recognition



- Goal:
  - Classify a new letter such that the probability of misclassification is minimized.

Bayes Decision Theory

- Concept 1: Priors (a priori probabilities) $\boxed{p(C_k)}$
  - What we can tell about the probability *before seeing the data*.
  - Example:

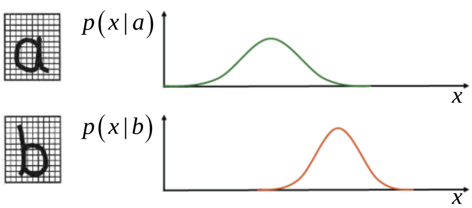


$$C_1 = a \qquad p(C_1) = 0.75$$
$$C_2 = b \qquad p(C_2) = 0.25$$

- In general: $\sum_k p(C_k) = 1$

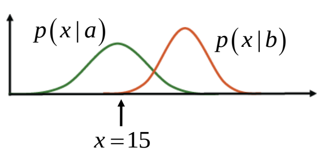
Bayes Decision Theory

- Concept 2: Conditional probabilities $\boxed{p(x|C_k)}$
  - Let x be a feature vector.
  - x measures/describes certain properties of the input.
    - E.g. number of black pixels, aspect ratio, …
  - $p(x|C_k)$ describes its likelihood for class C_k.



Bayes Decision Theory

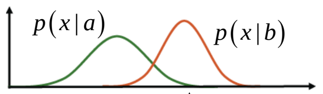
- Example:



- Question:
  - Which class?
  - Since $p(x|b)$ is much smaller than $p(x|a)$, the decision should be 'a' here.

Bayes Decision Theory

- **Example:**

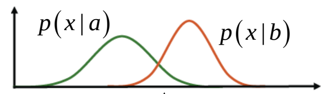


$x = 25$

- **Question:**
  - Which class?
  - Since $p(x|a)$ is much smaller than $p(x|b)$, the decision should be 'b' here.

Bayes Decision Theory

- **Example:**



$x = 20$

- **Question:**
  - Which class?
  - Remember that $p(a) = 0.75$ and $p(b) = 0.25$...
  - I.e., the decision should be again 'a'.
  - ⇒ How can we formalize this?

Bayes Decision Theory

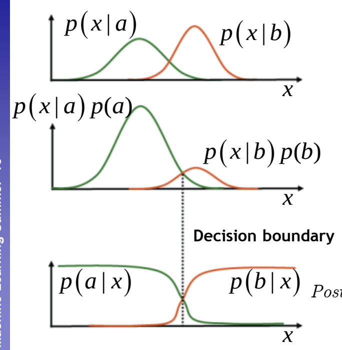
- **Concept 3: Posterior probabilities** $\boxed{p(C_k|x)}$
  - We are typically interested in the *a posteriori* probability, i.e. the probability of class C_k given the measurement vector x.
- **Bayes' Theorem:**

$$p(C_k|x) = \frac{p(x|C_k)\,p(C_k)}{p(x)} = \frac{p(x|C_k)\,p(C_k)}{\sum_i p(x|C_i)\,p(C_i)}$$

- **Interpretation**

$$Posterior = \frac{Likelihood \times Prior}{Normalization\ Factor}$$

Bayes Decision Theory



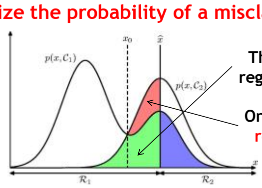
Likelihood

Likelihood $\times$ *Prior*

$$Posterior = \frac{Likelihood \times Prior}{NormalizationFactor}$$

Bayesian Decision Theory

- **Goal: Minimize the probability of a misclassification**



The green and blue regions stay constant.

Only the size of the red region varies!

$$\begin{aligned} p(\text{mistake}) &= p(\mathbf{x} \in \mathcal{R}_1, \mathcal{C}_2) + p(\mathbf{x} \in \mathcal{R}_2, \mathcal{C}_1) \\ &= \int_{\mathcal{R}_1} p(\mathbf{x}, \mathcal{C}_2)\,\mathrm{d}\mathbf{x} + \int_{\mathcal{R}_2} p(\mathbf{x}, \mathcal{C}_1)\,\mathrm{d}\mathbf{x}. \\ &= \int_{\mathcal{R}_1} p(\mathcal{C}_2|\mathbf{x})p(\mathbf{x})d\mathbf{x} + \int_{\mathcal{R}_2} p(\mathcal{C}_1|\mathbf{x})p(\mathbf{x})d\mathbf{x} \end{aligned}$$

Bayes Decision Theory

- **Optimal decision rule**
  - Decide for C_1 if

$$p(\mathcal{C}_1|x) > p(\mathcal{C}_2|x)$$

  - This is equivalent to

$$p(x|\mathcal{C}_1)p(\mathcal{C}_1) > p(x|\mathcal{C}_2)p(\mathcal{C}_2)$$

  - Which is again equivalent to (**Likelihood-Ratio test**)

$$\frac{p(x|\mathcal{C}_1)}{p(x|\mathcal{C}_2)} > \underbrace{\frac{p(\mathcal{C}_2)}{p(\mathcal{C}_1)}}_{\text{Decision threshold } \theta}$$

Generalization to More Than 2 Classes

- Decide for class k whenever it has the greatest posterior probability of all classes:

$$p(\mathcal{C}_k|x) > p(\mathcal{C}_j|x) \quad \forall j \neq k$$

$$p(x|\mathcal{C}_k)p(\mathcal{C}_k) > p(x|\mathcal{C}_j)p(\mathcal{C}_j) \quad \forall j \neq k$$

- Likelihood-ratio test

$$\frac{p(x|\mathcal{C}_k)}{p(x|\mathcal{C}_j)} > \frac{p(\mathcal{C}_j)}{p(\mathcal{C}_k)} \quad \forall j \neq k$$

Classifying with Loss Functions

- Generalization to decisions with a loss function
  - Differentiate between the possible decisions and the possible true classes.
  - Example: medical diagnosis
    - Decisions: *sick* or *healthy* (or: *further examination necessary*)
    - Classes: patient is *sick* or *healthy*
  - The cost may be asymmetric:

$$loss(decision = healthy|patient = sick) >> loss(decision = sick|patient = healthy)$$

Classifying with Loss Functions

- In general, we can formalize this by introducing a loss matrix L_{kj}

$$L_{kj} = loss\ for\ decision\ \mathcal{C}_j\ if\ truth\ is\ \mathcal{C}_k.$$

- Example: cancer diagnosis

$$L_{cancer\ diagnosis} = \begin{array}{c|cc} & \multicolumn{2}{c}{\text{Decision}} \\ \text{Truth} & \text{cancer} & \text{normal} \\ \hline \text{cancer} & 0 & 1000 \\ \text{normal} & 1 & 0 \end{array}$$

Classifying with Loss Functions

- Loss functions may be different for different actors.
  - Example:

$$\begin{array}{ccc} & \text{"invest"} & \text{"don't invest"} \end{array}$$

$$L_{stocktrader}(subprime) = \begin{pmatrix} -\frac{1}{2}c_{gain} & 0 \\ 0 & 0 \end{pmatrix}$$

$$L_{bank}(subprime) = \begin{pmatrix} -\frac{1}{2}c_{gain} & 0 \\ & 0 \end{pmatrix}$$

⇒ Different loss functions may lead to different Bayes optimal strategies.

Minimizing the Expected Loss

- Optimal solution is the one that minimizes the loss.
  - But: loss function depends on the true class, which is unknown.
- Solution: Minimize the expected loss

$$\mathbb{E}[L] = \sum_k \sum_j \int_{\mathcal{R}_j} L_{kj} p(\mathbf{x}, \mathcal{C}_k)\, \mathrm{d}\mathbf{x}$$

- This can be done by choosing the regions $\mathcal{R}_j$ such that

$$\mathbb{E}[L] = \sum_k L_{kj} p(\mathcal{C}_k|\mathbf{x})$$

which is easy to do once we know the posterior class probabilities $p(\mathcal{C}_k|\mathbf{x})$.

Minimizing the Expected Loss

- Example:
  - 2 Classes: C_1, C_2
  - 2 Decision: α_1, α_2
  - Loss function: $L(\alpha_j|\mathcal{C}_k) = L_{kj}$
  - Expected loss (= risk R) for the two decisions:

$$\mathbb{E}_{\alpha_1}[L] = R(\alpha_1|\mathbf{x}) = L_{11}p(\mathcal{C}_1|\mathbf{x}) + L_{21}p(\mathcal{C}_2|\mathbf{x})$$
$$\mathbb{E}_{\alpha_2}[L] = R(\alpha_2|\mathbf{x}) = L_{12}p(\mathcal{C}_1|\mathbf{x}) + L_{22}p(\mathcal{C}_2|\mathbf{x})$$

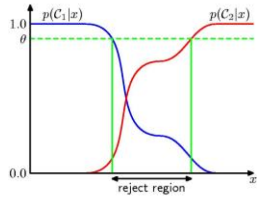
- Goal: Decide such that expected loss is minimized
  - I.e. decide α_1 if $R(\alpha_2|\mathbf{x}) > R(\alpha_1|\mathbf{x})$

Minimizing the Expected Loss

$$
\begin{aligned}
R(\alpha_2|\mathbf{x}) &> R(\alpha_1|\mathbf{x}) \\
L_{12}p(\mathcal{C}_1|\mathbf{x}) + L_{22}p(\mathcal{C}_2|\mathbf{x}) &> L_{11}p(\mathcal{C}_1|\mathbf{x}) + L_{21}p(\mathcal{C}_2|\mathbf{x}) \\
(L_{12} - L_{11})p(\mathcal{C}_1|\mathbf{x}) &> (L_{21} - L_{22})p(\mathcal{C}_2|\mathbf{x}) \\
\frac{(L_{12} - L_{11})}{(L_{21} - L_{22})} &> \frac{p(\mathcal{C}_2|\mathbf{x})}{p(\mathcal{C}_1|\mathbf{x})} = \frac{p(\mathbf{x}|\mathcal{C}_2)p(\mathcal{C}_2)}{p(\mathbf{x}|\mathcal{C}_1)p(\mathcal{C}_1)} \\
\frac{p(\mathbf{x}|\mathcal{C}_1)}{p(\mathbf{x}|\mathcal{C}_2)} &> \frac{(L_{21} - L_{22})}{(L_{12} - L_{11})}\frac{p(\mathcal{C}_2)}{p(\mathcal{C}_1)}
\end{aligned}
$$

⇒ Adapted decision rule taking into account the loss.

The Reject Option



- Classification errors arise from regions where the largest posterior probability $p(\mathcal{C}_k|\mathbf{x})$ is significantly less than 1.
  - These are the regions where we are relatively uncertain about class membership.
  - For some applications, it may be better to reject the automatic decision entirely in such a case and e.g. consult a human expert.

Discriminant Functions

- Formulate classification in terms of comparisons
  - Discriminant functions

$$y_1(x), \ldots, y_K(x)$$

  - Classify x as class C_k if

$$y_k(x) > y_j(x) \quad \forall j \neq k$$

- Examples (Bayes Decision Theory)

$$
\begin{aligned}
y_k(x) &= p(\mathcal{C}_k|x) \\
y_k(x) &= p(x|\mathcal{C}_k)p(\mathcal{C}_k) \\
y_k(x) &= \log p(x|\mathcal{C}_k) + \log p(\mathcal{C}_k)
\end{aligned}
$$

Different Views on the Decision Problem

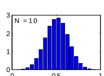
- $y_k(x) \propto p(x|\mathcal{C}_k)p(\mathcal{C}_k)$
  - First determine the class-conditional densities for each class individually and separately infer the prior class probabilities.
  - Then use Bayes' theorem to determine class membership.

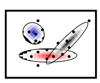
  ⇒ *Generative methods*
- $y_k(x) = p(\mathcal{C}_k|x)$
  - First solve the inference problem of determining the posterior class probabilities.
  - Then use decision theory to assign each new x to its class.

  ⇒ *Discriminative methods*
- Alternative
  - Directly find a discriminant function $y_k(x)$ which maps each input x directly onto a class label.

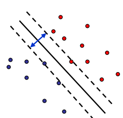
Next Lectures…

- Ways how to estimate the probability densities $p(x|\mathcal{C}_k)$
  - Non-parametric methods
    - Histograms
    - k-Nearest Neighbor
    - Kernel Density Estimation
  - Parametric methods
    - Gaussian distribution
    - Mixtures of Gaussians





- Discriminant functions
  - Linear discriminants
  - Support vector machines



⇒ ***Next lectures…***

References and Further Reading

- More information, including a short review of Probability theory and a good introduction in Bayes Decision Theory can be found in Chapters 1.1, 1.2 and 1.5 of

Christopher M. Bishop
Pattern Recognition and Machine Learning
Springer, 2006